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The vector generalization of the Landau–Lifshitz equation: Bäcklund transformation and solutions

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Abstract

The Bäcklund transformations (BTs) for vector generalization of the Landau–Lifshitz equation are presented. Also periodic and soliton-like solutions from BTs are computed.

Keywords: Bäcklund transformation; Vector evolution equations

1. Introduction

Recently Meshkov and Sokolov [1] have obtained a complete list of integrable equations in the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u}, \qquad (\boldsymbol{u}, \boldsymbol{u}) = 1, \tag{1}$$

on the sphere $\mathbb{S}^n$. Here $\boldsymbol{u} = \boldsymbol{u}(x, t)$ is an unknown vector function and f_i are scalar functions of variables $u_{[i,j]} = (\boldsymbol{u}_i, \boldsymbol{u}_j)$, $0 \leqslant i \leqslant j \leqslant 2$, $\boldsymbol{u}_i = \partial^i \boldsymbol{u}/\partial x^i$. Eq. (1) is invariant under motions in the configurational space with a metric $\mathrm{d}s^2 = (\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u})$; we therefore call it an isotropic equation. In the paper mentioned above a second scalar product $\langle\cdot,\cdot\rangle$ was introduced and the dependence f_i on $\tilde{u}_{[i,j]} = \langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle$ was considered. If f_i depend on both $u_{[i,j]}$ and $\tilde{u}_{[i,j]}$, the Eq. (1) is called an anisotropic one. We also refer to $u_{[i,j]}$ as isotropic variables and $\tilde{u}_{[i,j]}$ as anisotropic variables. Note that in this paper we interchange the roles of the brackets $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ in order to make the notation more convenient. In this article we use the same designations and in particular the abbreviation $\boldsymbol{u}^2 = (\boldsymbol{u}, \boldsymbol{u})$.

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Meshkov and Sokolov [1] presented as an example three anisotropic equations and one of them has the form

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}(u_{[1,1]} - \tilde{u}_{[0,0]})\boldsymbol{u}_1 + 3u_{[1,2]}\boldsymbol{u}. \tag{2}$$

Probably, these equations may be interesting for physical applications. As it is shown by Golubchik and Sokolov [2], if $\boldsymbol{u} \in \mathbb{S}^3$, then (2) is a higher symmetry of the Landau–Lifshitz equation

$$\boldsymbol{u}_t = \boldsymbol{u} \times \boldsymbol{u}_{xx} + \boldsymbol{u} \times (J\boldsymbol{u}), \qquad |\boldsymbol{u}| = 1, \tag{3}$$

where $\times$ denotes the usual vector product and the matrix $J = \operatorname{diag}(J_1, J_2, J_3)$ describes the anisotropy. In addition, this equation is particularly suitable for the study of periodic and quasiperiodic solutions of a classical nonlinear wave equation (like the sine-Gordon equation and the nonlinear Schrödinger equation) since it contains all these equations as special cases [3].

2. Bäcklund transformation

Meshkov and Sokolov [1] introduced auto-Bäcklund transformations (auto-BTs) for vectorial equations in the following form:

$$\boldsymbol{u}_x = f\boldsymbol{v} + g\boldsymbol{v}_x + h\boldsymbol{u}, \tag{4}$$

where f, g and h are scalar functions depending on

$$u_{[0,0]} = (\boldsymbol{u}, \boldsymbol{u}), \qquad v_{[i,j]} = (\boldsymbol{v}_i, \boldsymbol{v}_j), \qquad w_i = (\boldsymbol{u}, \boldsymbol{v}_i), \qquad 0 \leqslant i \leqslant j \leqslant 1. \tag{5}$$

In the anisotropic case f, g and h must also depend on additional variables

$$\tilde{u}_{[0,0]} = \langle \boldsymbol{u}, \boldsymbol{u} \rangle, \qquad \tilde{v}_{[i,j]} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle, \qquad \tilde{w}_i = \langle \boldsymbol{u}, \boldsymbol{v}_i \rangle, \qquad 0 \leqslant i \leqslant j \leqslant 1. \tag{6}$$

Differentiating (4) with respect to x we can express $\boldsymbol{u}_i$, $i \geqslant 1$, $u_{[i,j]}$ and $\tilde{u}_{[i,j]}$, $i \geqslant 0$, $j \geqslant 1$, $w_{[i,j]} = (\boldsymbol{u}_i, \boldsymbol{v}_j)$ and $\tilde{w}_{[i,j]} = \langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle$, $i \geqslant 1$, $j \geqslant 0$ through the independent variables $\boldsymbol{u}$, $\boldsymbol{v}_k$, $v_{[i,j]}$, $\tilde{v}_{[i,j]}$, w_k and $\tilde{w}_k$ with any i, k and $j \geqslant i$.

To find an auto-Bäcklund transformation for Eq. (1), we differentiate (4) with respect to t in virtue of (1). Then we exclude all dependent vector and scalar variables from the obtained equation. Splitting then this equation with respect to the independent variables that are not contained in f, g and h, we derive an overdetermined system of nonlinear partial differential equations for the functions f, g and h. If the system has a solution depending essentially on a parameter, this solution gives us the auto-Bäcklund transformation we are looking for.

Eq. (2) admits the following auto-Bäcklund transformation:

$$\frac{\boldsymbol{u}_x + \boldsymbol{v}_x}{2} = \frac{(\boldsymbol{u}, \boldsymbol{v}_x)(\boldsymbol{u} + \boldsymbol{v}) + f(\boldsymbol{u} + \boldsymbol{v}, \mu)(\boldsymbol{v} - (\boldsymbol{u}, \boldsymbol{v})\boldsymbol{u})}{(\boldsymbol{u} + \boldsymbol{v})^2}, \tag{7}$$

where $f^2(\boldsymbol{u}, \mu) = \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \mu \boldsymbol{u}^2$.

Prof. B. Fuchssteiner attracted our attention to the fact that transformation (7) for $n = 3$ should be the auto-BT for (3). With the help of direct calculations we have established that this observation is true.

3. Periodic and soliton-like solutions

Bäcklund transformations may be particularly applied for constructing soliton-like solutions. Below we attempt to find out some explicit solutions for the most interesting Eq. (2). We take for $\boldsymbol{v}$ a constant solution of (2) $\boldsymbol{v} = \boldsymbol{c}_1$, where $(\boldsymbol{c}_1, \boldsymbol{c}_1) = 1$. Taking into account that $v_{[0,i]} = 0,\ w_i = 0$ for $i > 0$, we obtain from (7)

$$\boldsymbol{u}_x = g\frac{\boldsymbol{c}_1 - w\boldsymbol{u}}{1+w}, \qquad g = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle + k + 2\tilde{w} + 2\mu(1+w)},$$
$$w = (\boldsymbol{u}, \boldsymbol{c}_1), \qquad \tilde{w} = \langle \boldsymbol{u}, \boldsymbol{c}_1\rangle, \qquad k = \langle \boldsymbol{c}_1, \boldsymbol{c}_1\rangle. \tag{8}$$

It follows from (8) that

$$w_{,x} = g(1-w). \tag{9}$$

If $w_{,x} = 0$, then two cases are possible: (i) $g \neq 0$, $w = 1$; (ii) $g = 0$, $\boldsymbol{u} = \boldsymbol{c}$. Case (ii) is not interesting. Case (i) gives the following equation: $\boldsymbol{u}_x = \frac{1}{2}(\boldsymbol{c}_1 - \boldsymbol{u})g$, and that implies $\boldsymbol{u}_x^2 = 0$. For a positive metric we have $\boldsymbol{u}_x = 0$ as in case (ii). Therefore the case $w_{,x} = 0$ is not interesting and we consider below that $w_{,x} \neq 0$.

Excluding g from (8) and (9) we obtain the linear equation

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}w} = \frac{\boldsymbol{c}_1 - w\boldsymbol{u}}{1-w^2},$$

having the following solution:

$$\boldsymbol{u} = \boldsymbol{c}_1 w + \boldsymbol{c}_2\sqrt{|1-w^2|}. \tag{10}$$

Here the vector $\boldsymbol{c}_2$ could be dependent on t, but we have found out that it is a constant. It follows from the conditions $\boldsymbol{u}^2 = \boldsymbol{c}_1^2 = 1$ and Eq. (10) that $(\boldsymbol{c}_1, \boldsymbol{c}_2) = 0$ and $\boldsymbol{c}_2^2 = \mathrm{signum}(1-w^2)$. We choose $\boldsymbol{c}_2^2 = 1$ because it is possible for both a positive and indefinite metric. This choice implies $w^2 \leqslant 1$. Notice that for a positive metric this inequality is a simple consequence of the Cauchy inequality $w^2 = (\boldsymbol{u}, \boldsymbol{c}_1)^2 \leqslant \boldsymbol{u}^2\boldsymbol{c}_1^2 = 1$.

Eq. (10) implies that $\langle \boldsymbol{u}, \boldsymbol{u}\rangle,\ \langle \boldsymbol{u}, \boldsymbol{c}_1\rangle$ and g are expressed via w only. Hence (9) is the only equation that we need. The following substitution $w = (z^2-1)/(z^2+1)$ simplifies (9) and we obtain

$$\boldsymbol{u} = \boldsymbol{c}_1\frac{z^2-1}{z^2+1} + \boldsymbol{c}_2\frac{2z}{z^2+1}, \tag{11}$$

$$z_x = \sqrt{\lambda z^2 + 2bz + a}, \tag{12}$$

where $\lambda = k + \mu,\ b = \langle \boldsymbol{c}_1, \boldsymbol{c}_2\rangle$ and $a = \langle \boldsymbol{c}_2, \boldsymbol{c}_2\rangle + \mu$.

Substituting (11) in system (2) and using (12) we find $z = z(x+\omega t)$, where $\omega = \mu - k/2$. So the speed of the wave $\boldsymbol{u}(x+\omega t)$ depends on both the Bäcklund parameter μ and the anisotropy parameter k. The properties of the function z depend on the values of λ and a and we have the following cases.

I. If $\lambda = 0$ then z is a quadratic or linear polynomial of $(x+\omega t)$. This case is not interesting and we omit explicit expressions.

II. If $\lambda = c^2 > 0$, then we have three subcases:

(a) If $b^2 - ac^2 < 0$, then setting $a = c^2(A^2+B^2),\ b = -c^2 B$ we obtain

$$z = A\sinh(c(x+\omega t)) + B; \tag{13}$$

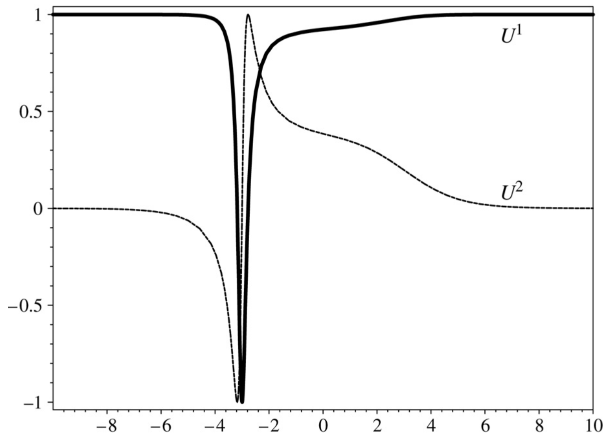


Fig. 1. Soliton-like solution (13) for $A = 0.5$, $B = 5$, $c = 1$.

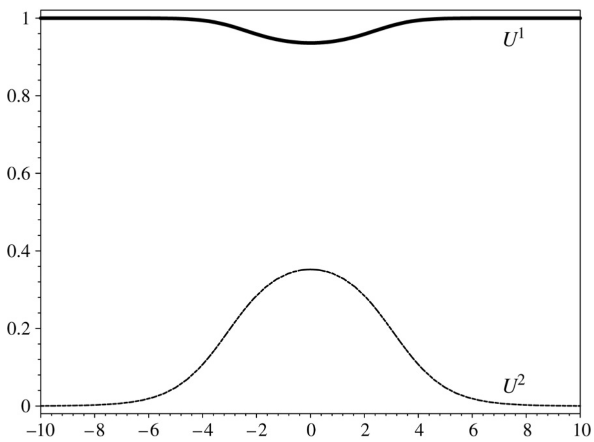


Fig. 2. Soliton solution (14) for $A = 0.3$, $B = 3$, $c = 1$.

(b) If $b^2 - ac^2 > 0$, then setting $a = c^2(B^2 - A^2),\ \ b = -c^2 B$ we obtain

$$z = A\cosh(c(x + \omega t)) + B; \tag{14}$$

(c) If $b^2 - ac^2 = 0$, then we obtain

$$z = \exp[c(x + \omega t)] - \frac{b}{c^2}. \tag{15}$$

III. If $\lambda = -c^2 < 0$, then the solution exists under the condition $b^2 + ac^2 > 0$ only and takes the following form:

$$z = A\sin(c(x + \omega t)) + B, \tag{16}$$

where $b = c^2 B$, $a = c^2(A^2 - B^2)$.

See Figs. 1–4 for the graphs of these solutions where $U^1 = (\boldsymbol{u}, \boldsymbol{c}_1) = w$ and $U^2 = (\boldsymbol{u}, \boldsymbol{c}_2) = \sqrt{1 - w^2}$ are the independent components of the vector $\boldsymbol{u}$.

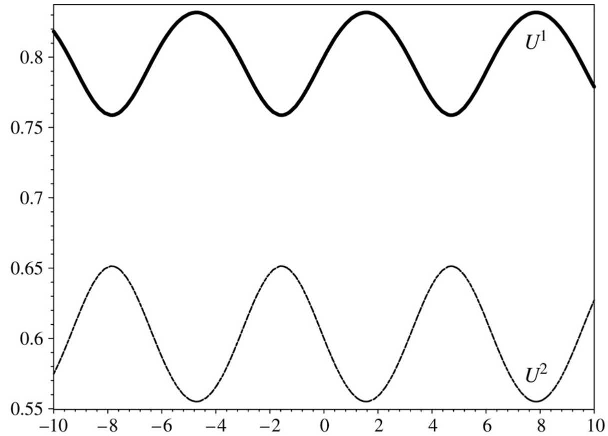


Fig. 3. Periodic solution (16) for $A = 0.1$, $B = 3$, $c = 1$.

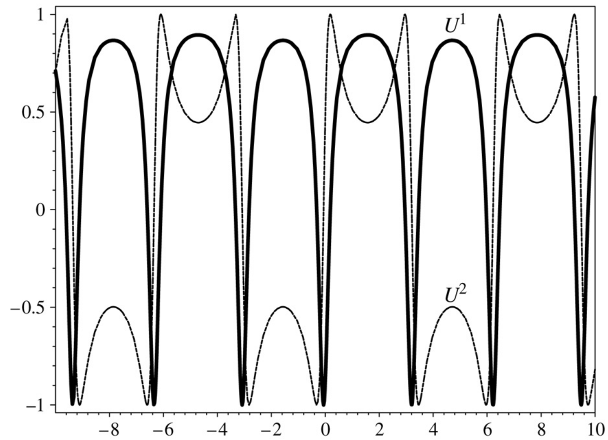


Fig. 4. Periodic solution (16) for $A = 4$, $B = 0.25$, $c = 1$.

Notice that the forms of the curves essentially depend on the values of the parameters A and B. For example, if $B = 0$, then in case (13) U^1 is an even function and U^2 is an odd one.

When the anisotropy vanishes we have in (12) $\lambda = a = \mu, b = 0$, hence $\mu \geqslant 0$. Setting $\mu = c^2$ we obtain for $c \neq 0$ a unique solution of (12), $z = \sinh(c(x + \omega t))$ and

$$U^1 = 2 \tanh c(x + \omega t), \qquad U^2 = 2 \sinh c(x + \omega t)(1 - \tanh^2 c(x + \omega t)).$$

These curves absolutely differ from the curves that are shown in Fig. 1.

Constructing more general solutions of (2) is a difficult task. Let $\boldsymbol{v}$ be a known solution. Then it seems clear that a new solution $\boldsymbol{u}$ of the Eq. (7) must contain an additional constant vector as a constant of integration. In the found solutions the new vector $\boldsymbol{c}_2$ is independent with $\boldsymbol{v} = \boldsymbol{c}_1$, but in the general case this is not true. When $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ is a Cartesian basis and

$$\boldsymbol{v} = \sum_{k=1}^{n} v^i \boldsymbol{c}_i, \qquad \boldsymbol{v}^i \neq 0, \ \forall i,$$

then the constant of integration must be a linear combination of the same vectors $\boldsymbol{c}_i$. We stress that Eq. (2) is integrable for any dimension of the configurational space. This implies that the equation possesses solutions in any subspaces. In particular, the solution (11) is situated on $\mathbb{S}^1$; solutions $\boldsymbol{u} \in \mathbb{S}^m \subset \mathbb{S}^n$ also exist, of course.

To find other solutions of Eq. (2) connected with (11) by the Bäcklund transformation we shall try to construct a superposition formula for them. We start from the usual assumption that the diagram

$$\begin{tikzcd} \boldsymbol{q} \arrow[r, "\mu"] & \boldsymbol{u} \\ \boldsymbol{v} \arrow[u, "\nu"] \arrow[r, "\mu"'] & \boldsymbol{p} \arrow[u, "\nu"'] \end{tikzcd} \tag{17}$$

for the Bäcklund transformations (7) is commutative (see [4], for example), where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q}$ are vector solutions of (2); μ and ν are parameters. This means that the equations

$$\begin{aligned} \frac{\boldsymbol{p}_x + \boldsymbol{v}_x}{2} &= \frac{(\boldsymbol{p}, \boldsymbol{v}_x)(\boldsymbol{p}+\boldsymbol{v}) + f(\boldsymbol{p}+\boldsymbol{v}, \mu)(\boldsymbol{v} - (\boldsymbol{p}, \boldsymbol{v})\boldsymbol{p})}{(\boldsymbol{p}+\boldsymbol{v})^2}, \\ \frac{\boldsymbol{q}_x + \boldsymbol{v}_x}{2} &= \frac{(\boldsymbol{q}, \boldsymbol{v}_x)(\boldsymbol{q}+\boldsymbol{v}) + f(\boldsymbol{q}+\boldsymbol{v}, \nu)(\boldsymbol{v} - (\boldsymbol{q}, \boldsymbol{v})\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{v})^2}, \\ \frac{\boldsymbol{u}_x + \boldsymbol{p}_x}{2} &= \frac{(\boldsymbol{u}, \boldsymbol{p}_x)(\boldsymbol{u}+\boldsymbol{p}) + f(\boldsymbol{u}+\boldsymbol{p}, \nu)(\boldsymbol{p} - (\boldsymbol{u}, \boldsymbol{p})\boldsymbol{u})}{(\boldsymbol{u}+\boldsymbol{p})^2}, \\ \frac{\boldsymbol{u}_x + \boldsymbol{q}_x}{2} &= \frac{(\boldsymbol{u}, \boldsymbol{q}_x)(\boldsymbol{u}+\boldsymbol{q}) + f(\boldsymbol{u}+\boldsymbol{q}\mu)(\boldsymbol{q} - (\boldsymbol{u}, \boldsymbol{q})\boldsymbol{u})}{(\boldsymbol{u}+\boldsymbol{q})^2}, \end{aligned} \tag{18}$$

hold, where $f^2(\boldsymbol{u}, \mu) = \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \mu \boldsymbol{u}^2$.

Adding all Eqs. (18) with different signs, we obtain the vector equation

$$g_1 \boldsymbol{u} + g_2 \boldsymbol{v} + g_3 \boldsymbol{p} + g_4 \boldsymbol{q} = 0, \tag{19}$$

with some scalar coefficients g_i depending on the first order variables $(\boldsymbol{p}, \boldsymbol{v}_x)$, $(\boldsymbol{q}, \boldsymbol{v}_x)$, $(\boldsymbol{u}, \boldsymbol{p}_x)$, $(\boldsymbol{u}, \boldsymbol{q}_x)$ and all zero order variables (5) and (6). Forming four scalar products of the expression (19) with the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}$ and $\boldsymbol{q}$, we obtain a system of four scalar equations. From the obtained equations we express three first order variables

$$\begin{aligned} (\boldsymbol{q}, \boldsymbol{v}_x) &= \frac{1 + (\boldsymbol{q}, \boldsymbol{v})}{1 + (\boldsymbol{p}, \boldsymbol{v})}(f(\boldsymbol{p}+\boldsymbol{v}, \mu) + (\boldsymbol{p}, \boldsymbol{v}_x)) - f(\boldsymbol{q}+\boldsymbol{v}, \nu), \\ (\boldsymbol{u}, \boldsymbol{q}_x) &= \frac{1 + (\boldsymbol{u}, \boldsymbol{q})}{1 + (\boldsymbol{q}, \boldsymbol{v})}((\boldsymbol{q}, \boldsymbol{v}_x) - (\boldsymbol{q}, \boldsymbol{v}) f(\boldsymbol{q}+\boldsymbol{v}, \nu)) - f(\boldsymbol{u}+\boldsymbol{q}, \mu), \\ (\boldsymbol{u}, \boldsymbol{p}_x) &= \frac{1 + (\boldsymbol{u}, \boldsymbol{p})}{1 + (\boldsymbol{p}, \boldsymbol{v})}((\boldsymbol{p}, \boldsymbol{v}_x) - (\boldsymbol{p}, \boldsymbol{v}) f(\boldsymbol{p}+\boldsymbol{v}, \mu)) - f(\boldsymbol{u}+\boldsymbol{p}, \nu) \end{aligned} \tag{20}$$

only and the fourth equation takes the following form:

$$f(\boldsymbol{p}+\boldsymbol{v}, \mu) + f(\boldsymbol{u}+\boldsymbol{p}, \nu) = f(\boldsymbol{q}+\boldsymbol{v}, \nu) + f(\boldsymbol{u}+\boldsymbol{q}, \mu). \tag{21}$$

Substituting (20) and (21) into (19) we obtain the identity. Eq. (21) is compatible with Eq. (2), hence it is the superposition formula that we looked for. For a vector equation the superposition formula is vectorial usually (see [5], for example) and the obtained result is unexpected. Nevertheless Eq. (21) is useful for searching the soliton-like solutions.

Lemma. *Let the diagram* (17) *be commutative. If Eq.* (2) *admits solutions* $\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q}$ *of the form*

$$\boldsymbol{v} = \frac{v^2-1}{v^2+1}\boldsymbol{c}_1 + \frac{2v}{v^2+1}\boldsymbol{c}_2, \qquad \boldsymbol{p} = \frac{p^2-1}{p^2+1}\boldsymbol{c}_1 + \frac{2p}{p^2+1}\boldsymbol{c}_2, \qquad \boldsymbol{q} = \frac{q^2-1}{q^2+1}\boldsymbol{c}_1 + \frac{2q}{q^2+1}\boldsymbol{c}_2, \tag{22}$$

where v, p *and* q *are some functions of* x *and* t, $\boldsymbol{c}_i$ *are unit vectors* $\boldsymbol{c}_i^2 = 1$ *and*

$$(\boldsymbol{c}_1, \boldsymbol{c}_2) = 0, \qquad \langle \boldsymbol{c}_1, \boldsymbol{c}_1 \rangle = k_1, \qquad \langle \boldsymbol{c}_2, \boldsymbol{c}_2 \rangle = k_2, \qquad \langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle = k_3.$$

Then solution $\boldsymbol{u}$ *takes the form*

$$\boldsymbol{u} = \frac{u^2-1}{u^2+1}\boldsymbol{c}_1 + \frac{2u}{u^2+1}\boldsymbol{c}_2.$$

Proof. As Eq. (19) does not contain the vector $\boldsymbol{u}_x$ and the vectors $\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q}$ depend on the vectors $\boldsymbol{c}_1, \boldsymbol{c}_2$, then $\boldsymbol{u}$ will be a function of two vectors $\boldsymbol{c}_1, \boldsymbol{c}_2$. If the vector $\boldsymbol{u}$ depends on some constant vector $\boldsymbol{c}_3$, then from Eq. (19) it follows that $\boldsymbol{c}_3$ can be expressed through $\boldsymbol{c}_1, \boldsymbol{c}_2$. The final form of $\boldsymbol{u}$ determined from the condition $(\boldsymbol{u}, \boldsymbol{u}) = 1$.

According to the lemma Eq. (21) takes the form

$$\begin{aligned} &\frac{(pv+1)\sqrt{\mu(p^2+1)(v^2+1) + k_1(pv-1)^2 + k_2(p+v)^2 + 2k_3(p+v)(pv-1)}}{(v^2+1)(p^2+1)} \\ &\quad + \frac{(pu+1)\sqrt{\nu(p^2+1)(u^2+1) + k_1(pu-1)^2 + k_2(p+u)^2 + 2k_3(p+u)(pu-1)}}{(u^2+1)(p^2+1)} \\ &= \frac{(qv+1)\sqrt{\nu(q^2+1)(v^2+1) + k_1(qv-1)^2 + k_2(q+v)^2 + 2k_3(q+v)(qv-1)}}{(v^2+1)(q^2+1)} \\ &\quad + \frac{(qu+1)\sqrt{\mu(q^2+1)(u^2+1) + k_1(qu-1)^2 + k_2(q+u)^2 + 2k_3(q+u)(qu-1)}}{(u^2+1)(q^2+1)}. \end{aligned} \tag{23}$$

The presented expression is too cumbersome, so we consider the special case $k_2 = k_1$ and $k_3 = 0$. For simplification we pass to new variables α, β, γ and δ with the help of the equations $\sqrt{1+u^2} = u + \alpha$, $\sqrt{1+p^2} = p + \beta$, $\sqrt{1+q^2} = q + \gamma$, $\sqrt{1+v^2} = v + \delta$, then we obtain from (23)

$$\alpha = \frac{\delta(1+\beta\gamma)(c_1-c_2) - (\beta-\gamma)(c_1+c_2)}{(1+\beta\gamma)(c_1-c_2) + \delta(\beta-\gamma)(c_1+c_2)}, \qquad c_1^2 = \mu + k_1, \qquad c_2^2 = \nu + k_1.$$

As it was shown if $v = 0$, $p = \sinh(c_1(x + \omega_1 t))$, $q = \sinh(c_2(x + \omega_2 t))$, $\omega_1 = c_1^2 - 3k_1/2$, $\omega_2 = c_2^2 - 3k_1/2$, $\delta = 1$ and

$$\begin{aligned} &\alpha = \frac{(c_1-c_2)(1+\beta\gamma) - (c_1+c_2)(\beta-\gamma)}{(c_1-c_2)(1+\beta\gamma) + (c_1+c_2)(\beta-\gamma)}, \qquad \beta = \exp[-c_1(x+\omega_1 t)], \\ &\quad \gamma = \exp[-c_2(x+\omega_2 t)], \end{aligned} \tag{24}$$

finally for vector $\boldsymbol{u}$ we have

$$\boldsymbol{u} = \frac{(\alpha^2+2\alpha-1)(\alpha^2-2\alpha-1)}{(1+\alpha^2)^2}\boldsymbol{c}_1 + \frac{4\alpha(1-\alpha^2)}{(1+\alpha^2)^2}\boldsymbol{c}_2. \tag{25}$$

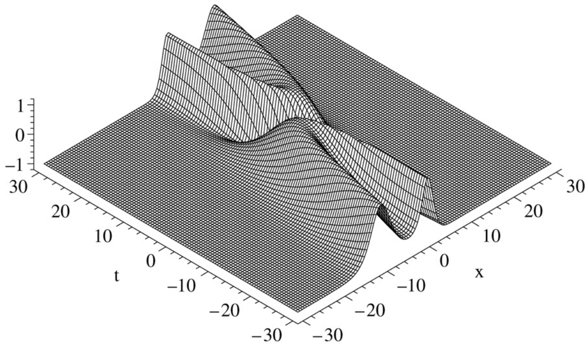


Fig. 5. Two soliton solution U^1.

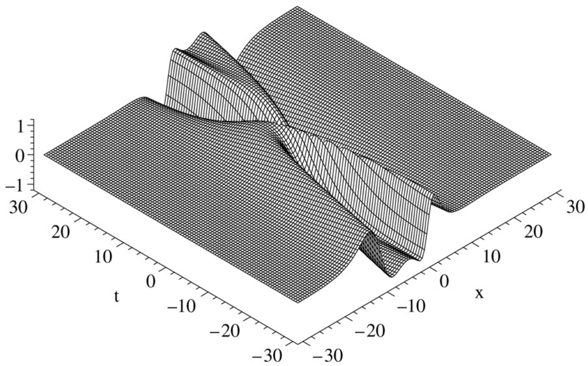
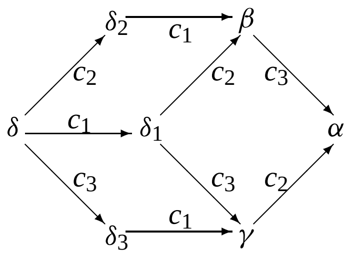


Fig. 6. Two soliton solution U^2.

See Figs. 5 and 6 for the graphs of these solutions where $U^1 = (\boldsymbol{u}, \boldsymbol{c}_1)$, $U^2 = (\boldsymbol{u}, \boldsymbol{c}_2)$ are the independent components of the vector $\boldsymbol{u}$ and $k_1 = 0.2$, $c_1 = 1/3$, $c_2 = 1/2$.

Similarly for next solution we have



where $\delta = 1$, $\delta_i = \exp[-c_i(x + \omega_i t)]$, $i = 1, 2, 3$ and

$$\begin{aligned}
\alpha &= \frac{\delta_1(1+\beta\gamma)(c_3-c_2) - (\gamma-\beta)(c_3+c_2)}{(1+\beta\gamma)(c_3-c_2) + \delta_1(\gamma-\beta)(c_3+c_2)},\\
\beta &= \frac{(c_1-c_2)(1+\delta_1\delta_2) - (c_1+c_2)(\delta_1-\delta_2)}{(c_1-c_2)(1+\delta_1\delta_2) + (c_1+c_2)(\delta_1-\delta_2)},\\
\gamma &= \frac{(c_3-c_1)(1+\delta_3\delta_1) - (c_3+c_1)(\delta_3-\delta_1)}{(c_3-c_1)(1+\delta_3\delta_1) + (c_3+c_1)(\delta_3-\delta_1)}.
\end{aligned} \tag{26}$$

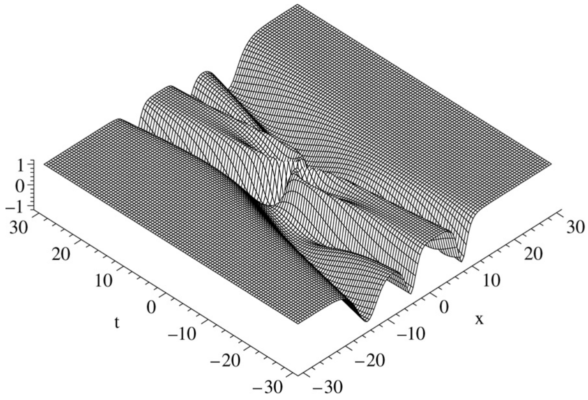


Fig. 7. Three soliton solution U^1.

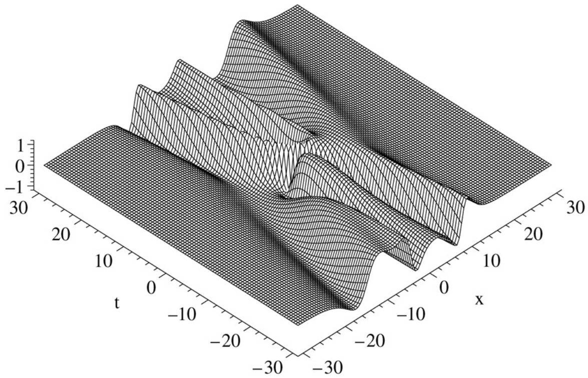


Fig. 8. Three soliton solution U^2.

See Figs. 7 and 8 for the graphs of solutions (25) and (26) for $k_1 = 0.2$, $c_1 = 1/2$, $c_2 = 2/3$, $c_3 = 1/3$ where U^1 and U^2 are the independent components of the vector $\boldsymbol{u}$.

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